SUBJECT: MATHEMATICS

PAPER: ADVANCED MATHEMATICAL METHODS
PAPER CODE:H 3051(PAPER III)
CLASS: MSc III SEMESTER

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Q1-04, Pdf. $\frac{08}{02}$ (01) H-3051, Dr. Sahish Kumar

(4) Using Hilbert-Schmidt theorem, solve

$$\phi(x) = x + \lambda \int_0^{2\pi} \sin(x+\xi)\, \phi(\xi)\, d\xi \qquad \text{——— (1)}$$

Solution. Given equation is

$$\phi(x) = x + \lambda \int_0^{2\pi} \sin(x+\xi)\, \phi(\xi)\, d\xi \qquad \text{——— (2)}$$

The homogeneous equation is

$$\phi(x) = \lambda \int_0^{2\pi} \sin(x+\xi)\, \phi(\xi)\, d\xi \qquad \text{——— (3)}$$

$$\phi(x) = \lambda \int_0^{2\pi} (\sin x \cos\xi + \cos x \sin\xi)\, \phi(\xi)\, d\xi \qquad \text{——— (4)}$$

$$\phi(x) = \lambda \sin x \cdot c_1 + \lambda \cos x \cdot c_2 \qquad \text{——— (5)}$$

where

$$c_1 = \int_0^{2\pi} \cos\xi\, \phi(\xi)\, d\xi \qquad \text{——— (6)}$$

$$c_2 = \int_0^{2\pi} \sin\xi\, \phi(\xi)\, d\xi \qquad \text{——— (7)}$$

From, (5), we have

$$\phi(\xi) = \lambda c_1 \sin\xi + \lambda c_2 \cos\xi$$

Equations (6) and (7) ⇒

$$c_1 = \int_0^{2\pi} \cos\xi \left[\lambda c_1 \sin\xi + \lambda c_2 \cos\xi\right] d\xi$$

$$\Rightarrow c_1 = \lambda c_1 \int_0^{2\pi} \sin\xi \cos\xi\, d\xi + \lambda c_2 \int_0^{2\pi} \cos^2\xi\, d\xi$$

$$= 0 + \lambda c_2 \int_0^{2\pi} \frac{1+\cos 2\xi}{2}\, d\xi$$

$$c_1 = \lambda c_2 \cdot \frac{1}{2} \cdot 2\pi$$

$$\Rightarrow c_1 - \frac{2\pi\lambda}{2} c_2 = 0 \Rightarrow c_1 - \pi\lambda c_2 = 0 \qquad \text{— (8)}$$

Eqn's (6) and (7) ⇒ $c_2 = \int_0^{2\pi} \sin\xi \left[\lambda c_1 \sin\xi + \lambda c_2 \cos\xi\right] d\xi$

$$\Rightarrow c_2 = \lambda c_1 \int_0^{2\pi} \sin^2\xi\, d\xi + \lambda c_2 \int_0^{2\pi} \sin\xi \cos\xi\, d\xi$$

$$\Rightarrow c_2 = \lambda c_1 \cdot \int_0^{2\pi} \frac{1-\cos 2\xi}{2}\, d\xi + \lambda c_2 \cdot 0 = 0$$

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$$c_2 = \frac{\lambda c_1}{2}\int_0^{2\pi} d\xi - \frac{\lambda c_1}{2}\int_0^{2\pi} \cos 2\xi \, d\xi$$

$$c_2 = \pi\lambda c_1$$

$$\Rightarrow -\pi\lambda c_1 + c_2 = 0 \quad \text{———} \quad (9)$$

The characteristic polynomial in λ of (8) & (9) is given by

$$D(\lambda) = \begin{vmatrix} 1 & -\pi\lambda \\ -\pi\lambda & 1 \end{vmatrix} \Rightarrow D(\lambda) = 1 - \pi^2\lambda^2$$

To find eigen values of (3), put $D(\lambda) = 0$

$$\Rightarrow 1 - \pi^2\lambda^2 = 0 \Rightarrow \lambda = \pm\frac{1}{\pi} \Rightarrow \lambda_1 = \frac{1}{\pi}$$

Note: if $D(\lambda) \neq 0$ then soln of (1) is $\phi(x) = x$ $[c_1 = 0 = c_2]$ and soln of (2) is $\phi(x) = 0$ $\quad \lambda_2 = -\frac{1}{\pi}$

Case II To find eigen function corresponding to eigen value

$$\lambda_1 = \frac{1}{\pi}$$

The system (8) & (9) $\Rightarrow c_1 - \pi\lambda_1 c_2 = 0$

$$\Rightarrow c_1 - \pi \times \frac{1}{\pi} c_2 = 0 \Rightarrow c_1 - c_2 = 0$$

$$\Rightarrow c_1 = c_2 \quad \text{—} (10)$$

and $-\pi\lambda_1 c_1 + c_2 = 0$

$$\Rightarrow -\pi \times \frac{1}{\pi} c_1 + c_2 = 0 \Rightarrow c_1 - c_2 = 0 \quad \text{—} (11)$$

$\therefore$ (10) & (11) $\Rightarrow c_1 = c_2$

put $c_1 = c_2$ in (4), we get

$$\phi(x) = \lambda_1 \sin x \cdot c_2 + \lambda_1 \cos x \, c_2$$

$$\phi(x) = \frac{1}{\pi} c_2 [\sin x + \cos x]$$

$$\Rightarrow \phi_1(x) = \sin x + \cos x \quad \text{if we take } \frac{c_2}{\pi} = 1$$

$$\Phi_1(x) = \frac{\sin x + \cos x}{\|\sin x + \cos x\|}$$

$$\|\sin x + \cos x\| = \sqrt{\int_0^{2\pi} (\sin^2 x + \cos^2 x + \sin 2x)\, dx} = \left[\int_0^{2\pi} 1 \cdot dx + \int_0^{2\pi} \sin 2x \, dx\right]^{\frac{1}{2}}$$

$$\|\sin x + \cos x\| = \sqrt{2\pi} \quad \left[\because \int_0^{2\pi} \sin 2x \, dx = 0\right]$$

ii $\phi_1(x) = \dfrac{\sin x + \cos x}{\sqrt{2\pi}}$ ——— (12)

$$F_1 = \int_0^{2\pi} f(x)\, \phi_1(x)\, dx$$

$$= \int_0^{2\pi} x \cdot \left[\frac{\sin x + \cos x}{\sqrt{2\pi}}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_0^{2\pi} x \sin x\, dx + \int_0^{2\pi} x \cos x\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[-x \cdot \cos x \Big|_0^{2\pi} + \int_0^{2\pi} \cos x\, dx + x \sin x \Big|_0^{2\pi} - \int_0^{2\pi} \sin x\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[-2\pi + 0 + 0 + [\cos x]_0^{2\pi}\right]$$

$$F_1 = -\sqrt{2\pi}$$

Case III

To find eigen function corresponding to eigen value $\lambda_2 = -\frac{1}{\pi}$

system (8) & (9) $\Rightarrow$

$c_1 - \pi \lambda_2 c_2 = 0 \Rightarrow c_1 + c_2 = 0$ —(13)

and

$-\pi \lambda_2 c_1 + c_2 = 0$ ——— (14)

$\Rightarrow c_1 + c_2 = 0$

from (13) & (14), we get

$c_1 = -c_2$ or $c_2 = -c_1$

So put $c_1 = -c_2$ into (4), we have

$\phi(x) = \lambda_2 \sin x\, c_1 - \lambda_2 \cos x\, c_1$

$\phi(x) = \lambda_2 c_1 [\sin x - \cos x]$

$\Rightarrow \phi_2(x) = (\sin x - \cos x)$ if we take $\lambda_2 c_1 = 1$, $\lambda_2 = -\frac{1}{\pi}$

$$\Phi_2(x) = \frac{\sin x - \cos x}{\|\sin x - \cos x\|}$$

$$\|\sin x - \cos x\| = \sqrt{\int_0^{2\pi} (\sin^2 x + \cos^2 x - \sin 2x)\,dx}$$

$$= \sqrt{\int_0^{2\pi} dx - \int_0^{2\pi} \sin 2x\,dx} = \sqrt{2\pi}$$

$$\therefore \Phi_2(x) = \frac{\sin x - \cos x}{\sqrt{2\pi}} \quad \text{—— (15)}$$

$$f_2 = \int_0^{2\pi} f(x)\,\Phi_2(x)\,dx = \int_0^{2\pi} x.\frac{\sin x - \cos x}{\sqrt{2\pi}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_0^{2\pi} x \sin x\,dx - \int_0^{2\pi} x \cos x\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[-x\cos x\Big|_0^{2\pi} + \int_0^{2\pi}\cos x\,dx - \left\{x \sin x\Big|_0^{2\pi} - \int_0^{2\pi} \sin x\,dx\right\}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[-2\pi\cos 2\pi + 0 + 0 - 2\pi \sin 2\pi + 0\right]$$

$$f_2 = -\sqrt{2\pi}$$

$\therefore$ Required solution of given equation (1) is given by

$$\phi(x) = f(x) + \frac{\lambda}{\lambda_1 - \lambda} f_1 \Phi_1(x) + \frac{\lambda}{\lambda_2 - \lambda} f_2 \Phi_2(x) \quad \text{—— (16)}$$

$$\phi(x) = x + \frac{\lambda}{\frac{1}{\pi} - \lambda}(-\sqrt{2\pi})\left(\frac{\sin x + \cos x}{\sqrt{2\pi}}\right) \qquad \left(\begin{array}{l}\text{if } \lambda \neq \lambda_1 \\ \lambda \neq \lambda_2\end{array}\right)$$

$$+ \frac{\lambda}{-\frac{1}{\pi} - \lambda}\cdot(-\sqrt{2\pi})\left[\frac{\sin x - \cos x}{\sqrt{2\pi}}\right] \quad \text{—— (17)}$$

$$\phi(x) = x - \frac{\pi\lambda}{1 - \pi\lambda}(\sin x + \cos x) + \frac{\pi\lambda}{1 + \pi\lambda}\cdot(\sin x - \cos x)$$

$$\phi(x) = x + \pi\lambda \sin x\left[\frac{1}{1+\pi\lambda} - \frac{1}{1-\pi\lambda}\right] - \pi\lambda\cos x\left[\frac{1}{1+\pi\lambda} + \frac{1}{1-\pi\lambda}\right]$$

Ans.

Note, if $\lambda = \lambda_1$ or $\lambda = \lambda_2$, $f_1 \neq 0$, $f_2 \neq 0$.
Equation (1) has no solution.

(5) Using Hilbert schmidt theorem, prove

$$\phi(x) = \cos 3x + \lambda \int_0^{\pi} \cos(x+\xi)\,\phi(\xi)\,d\xi$$

Solution. Given

$$\phi(x) = \cos 3x + \lambda \int_0^{\pi} \cos(x+\xi)\,\phi(\xi)\,d\xi \quad \text{—— (1)}$$

Its homogeneous equation is

$$\phi(x) = \lambda \int_0^{\pi} \cos(x+\xi)\,\phi(\xi)\,d\xi \quad \text{—— (2)}$$

$$\phi(x) = \lambda \int_0^{\pi} (\cos x \cos\xi - \sin x \sin\xi)\,\phi(\xi)\,d\xi$$

$$\Rightarrow \phi(x) = \lambda \cos x \cdot c_1 - \lambda \sin x \cdot c_2 \quad \text{—— (3)}$$

where

$$c_1 = \int_0^{\pi} \cos\xi\,\phi(\xi)\,d\xi \quad \text{—— (4)}$$

$$c_2 = \int_0^{\pi} \sin\xi\,\phi(\xi)\,d\xi \quad \text{—— (5)}$$

from (3), $\phi(\xi) = \lambda c_1 \cos\xi - \lambda c_2 \sin\xi$ —— (6)

From (4) and (6), we have

$$c_1 = \int_0^{\pi} \cos\xi\,[\lambda c_1 \cos\xi - \lambda c_2 \sin\xi]\,d\xi$$

$$c_1 = \lambda c_1 \int_0^{\pi} \cos^2\xi\,d\xi - \lambda c_2 \int_0^{\pi} \sin\xi \cos\xi\,d\xi$$

$$c_1 = \lambda c_1 \int_0^{\pi} \left(\frac{1+\cos 2\xi}{2}\right) d\xi - \lambda c_2 \left(\frac{\sin^2\xi}{2}\right)_0^{\pi}$$

$$= \frac{\lambda c_1}{2}\int_0^{\pi} d\xi + \frac{\lambda c_1}{2}\int_0^{\pi} \cos 2\xi\,d\xi - 0 c_2$$

$$c_1 = \frac{\pi\lambda}{2} c_1 + 0 - 0 c_2$$

$$\Rightarrow \left(1 - \frac{\pi\lambda}{2}\right) c_1 + 0 c_2 = 0 \quad \text{—— (7)}$$

From (5) and (6), we have

$$c_2 = \int_0^{\pi} \sin\xi\,[\lambda c_1 \cos\xi - \lambda c_2 \sin\xi]\,d\xi$$

$$= \lambda c_1 \int_0^{\pi} \sin\xi \cos\xi\,d\xi - \lambda c_2 \int_0^{\pi} \frac{1-\cos 2\xi}{2}\,d\xi$$

$$c_2 = \frac{\lambda c_1}{2}\left[\sin^2\xi\right]_0^{\pi} - \lambda c_2 \cdot \frac{\pi}{2} + 0$$

$$c_2 = 0c_1 + \frac{\pi\lambda}{2} c_2$$

$$\Rightarrow 0c_1 + \left(1+\frac{\pi\lambda}{2}\right)c_2 = 0 \quad \text{——— (8)}$$

The characteristic polynomial of (7) and (8) is

$$D(\lambda) = \begin{vmatrix} 1-\frac{\pi\lambda}{2} & 0 \\ 0 & 1+\frac{\pi\lambda}{2} \end{vmatrix} = \left(1-\frac{\pi^2\lambda^2}{4}\right)$$

To find eigen values of (2), put $D(\lambda)=0$

$$\Rightarrow 1-\frac{\pi^2\lambda^2}{4} = 0 \Rightarrow \lambda^2 = \frac{4}{\pi^2} \Rightarrow \lambda_1 = \frac{2}{\pi}$$

and $\lambda_2 = -\frac{2}{\pi}$

Case I. If $D(\lambda) \neq 0$ then $c_1 = 0 = c_2$

$\therefore \phi(x) = \lambda \cos x\, c_1 - \lambda \sin x \cdot c_2$

$\Rightarrow \phi(x) = 0$

$\therefore$ solution of (1) is $\phi(x) = \cos 3x$ ——— (9)

and solution of (2) is $\phi(x) = 0$. ——— (10)

Case II. If $D(\lambda)=0$

Take $\lambda_1 = \frac{2}{\pi}$

To find eigen vector corresponding to $\lambda_1 = \frac{2}{\pi}$

system [(7) & (8)] $\Rightarrow \left(1-\frac{\pi\lambda_1}{2}\right)c_1 = 0 \Rightarrow \left(1-\frac{\pi}{2}\times\frac{2}{\pi}\right)c_1 = 0$

$\Rightarrow 0.c_1 = 0 \Rightarrow c_1$ is arbitrary

and $\left(1+\frac{\pi\lambda_1}{2}\right)c_2 = 0 \Rightarrow 2.c_2 = 0 \Rightarrow c_2 = 0$

put $c_2 = 0$ into (3), we have

$\phi(x) = \lambda_1 \cos x\, c_1 - \lambda_1 \sin x \cdot c_2$

$\phi(x) = \lambda_1 c_1 \cos x \quad [\because c_2 = 0]$

$\Rightarrow \phi_1(x) = \cos x$ if $\lambda_1 c_1 = 1$, $\lambda_1 = \frac{2}{\pi}$

$\bar{\phi}_1(x) = \frac{\cos x}{\|\cos x\|}$, $\|\cos x\| = \sqrt{\int_0^{\pi} \cos^2 x\, dx} = \sqrt{\int_0^{\pi} \frac{1+\cos 2x}{2} dx}$

$\bar{\phi}_1(x) = \sqrt{\frac{2}{\pi}} \cos x$ ——— (11) $\quad \|\cos x\| = \sqrt{\frac{\pi}{2}}$

$$F_1 = \int_0^{\pi} f(x)\,\Phi_1(x)\,dx$$

$$= \int_0^{\pi} \cos 3x . \cos x \left(\sqrt{\tfrac{2}{\pi}}\right) dx$$

$$= \sqrt{\tfrac{2}{\pi}} \int_0^{\pi} \frac{2\cos 3x \cos x}{2}\, dx$$

$$= \tfrac{1}{2}\sqrt{\tfrac{2}{\pi}} \int_0^{\pi} (\cos 4x + \cos 2x)\, dx$$

$$F_1 = 0$$

$\therefore$ if $\lambda \neq \lambda_1$, $F_1 = 0$ Then system (1) has infinitely many solutions

Case III if $D(\lambda) = 0$, Take $\lambda_2 = -\frac{2}{\pi}$

system ((7) & (8)) $\Rightarrow$

$$\left(1 - \frac{\pi\lambda}{2}\right) c_1 = 0 \Rightarrow (1+1)\, c_1 = 0 \Rightarrow c_1 = 0$$

$$\left(1 + \frac{\pi\lambda_2}{2}\right) c_2 = 0 \Rightarrow 0 . c_2 = 0 \Rightarrow c_2 \text{ is arbitrary}$$

put $c_1 = 0$ into (3), we have

$$\phi(x) = -\lambda_2 c_2 . \sin x \quad \text{if we take } -\left(-\tfrac{2}{\pi}\right) c_2 = 1$$

$$\Rightarrow \phi_2(x) = \sin x \qquad \text{ie } \tfrac{2}{\pi} c_2 = 1$$

Now,

$$\Phi_2(x) = \frac{\sin x}{\|\sin x\|}, \quad \|\sin x\| = \sqrt{\int_0^{\pi} \sin^2 x\, dx}$$

$$\Phi_2(x) = \sqrt{\tfrac{2}{\pi}} \sin x \qquad = \sqrt{\tfrac{\pi}{2}}$$

$$F_2 = \int_0^{\pi} f(x)\,\Phi_2(x)\,dx = \int_0^{\pi} \cos 3x . \sqrt{\tfrac{2}{\pi}} \sin x\, dx$$

$$F_2 = \sqrt{\tfrac{2}{\pi}} . \tfrac{1}{2} \int_0^{\pi} 2\cos 3x \sin x\, dx = \tfrac{1}{2}\sqrt{\tfrac{2}{\pi}} \int_0^{\pi} (\sin 4x - \sin 2x)\, dx$$

$$\Rightarrow F_2 = 0 .$$

If $\lambda \neq \lambda_2$, $F_2 = 0$ then eqn (1) has infinitely many solutions.

$\therefore$ Required solu is given by

$$\phi(x) = \cos 3x + A \cos x + B \sin x$$

, A & B are arbitrary constants.

(6) Using Hilbert schmidt theorem, Solve

$$\phi(x) = 1 + \lambda \int_0^{\pi} \cos(x+\xi)\, \phi(\xi)\, d\xi$$

Solution. Given

$$\phi(x) = 1 + \lambda \int_0^{\pi} \cos(x+\xi)\, \phi(\xi)\, d\xi \qquad \text{—— (1)}$$

The homogeneous eqn of (1) is given by

$$\phi(x) = \lambda \int_0^{\pi} (\cos x \cos\xi - \sin x \sin\xi)\, \phi(\xi)\, d\xi \qquad -(2)$$

$$\Rightarrow \quad \phi(x) = \lambda \cos x \cdot C_1 - \lambda \sin x \cdot C_2 \qquad \text{—— (3)}$$

where

$$C_1 = \int_0^{\pi} \cos\xi\, \phi(\xi)\, d\xi \qquad \text{—— (4)}$$

$$C_2 = \int_0^{\pi} \sin\xi\, \phi(\xi)\, d\xi \qquad \text{—— (5)}$$

From (2),

$$\phi(\xi) = \lambda C_1 \cos\xi - \lambda C_2 \sin\xi \qquad \text{—— (6)}$$

Equations (4) and (6) $\Rightarrow$

$$C_1 = \int_0^{\pi} \cos\xi \left[\lambda C_1 \cos\xi - \lambda C_2 \sin\xi\right] d\xi$$

$$C_1 = \lambda C_1 \int_0^{\pi} \cos^2\xi\, d\xi - \lambda C_2 \int_0^{\pi} \sin\xi \cos\xi\, d\xi$$

$$C_1 = \lambda C_1 \int_0^{\pi} \left(\frac{1+\cos 2\xi}{2}\right) d\xi - \lambda C_2 \left[\frac{\sin^2\xi}{2}\right]_0^{\pi}$$

$$C_1 = \frac{\lambda C_1}{2} \int_0^{\pi} d\xi + \frac{\lambda C_1}{2} \int_0^{\pi} \cos 2\xi\, d\xi - 0 \cdot C_2$$

$$C_1 = \frac{\pi\lambda}{2} C_1 + 0 - 0\, C_2 \qquad \text{—— (7)}$$

$$\Rightarrow \quad \left(1 - \frac{\pi\lambda}{2}\right) C_1 = 0$$

Equations (5) and (6) $\Rightarrow$

$$C_2 = \int_0^{\pi} \sin\xi \left[\lambda C_1 \cos\xi - \lambda C_2 \sin\xi\right] d\xi$$

$$C_2 = \lambda C_1 \int_0^{\pi} \sin\xi \cos\xi\, d\xi - \lambda C_2 \int_0^{\pi} \sin^2\xi\, d\xi$$

$$c_2 = \lambda c_1 \left[\frac{-\sin^2 \xi}{2} \right]_0^{\pi} - \frac{\lambda c_2}{2} \int_0^{\pi} (1 - \cos 2\xi)\, d\xi$$

$$= 0 \cdot c_1 - \frac{\lambda c_2}{2} \left[\int_0^{\pi} d\xi - \int_0^{\pi} \cos 2\xi \, d\xi \right]$$

$$c_2 = 0 \cdot c_1 - \frac{\pi \lambda}{2} c_2 + 0$$

$$\Rightarrow 0 c_1 + \left(1 + \frac{\pi\lambda}{2}\right) c_2 = 0 \qquad \text{--------- (8)}$$

The characteristic polynomial of the system [(7), (8)] is

$$D(\lambda) = \begin{vmatrix} 1 - \frac{\pi\lambda}{2} & 0 \\ 0 & \frac{1 + \pi\lambda}{2} \end{vmatrix} = \left(1 - \frac{\pi^2 \lambda^2}{4}\right)$$

To find eigen values, put $D(\lambda) = 0$

$$\Rightarrow 1 - \frac{\pi^2 \lambda^2}{4} = 0 \Rightarrow \lambda^2 = \frac{4}{\pi^2} \Rightarrow \lambda = \frac{2}{\pi}, -\frac{2}{\pi}$$

Let $\lambda_1 = \frac{2}{\pi}$, $\lambda_2 = -\frac{2}{\pi}$

To find eigen vector corresponding to $\lambda_1 = \frac{2}{\pi}$

Equ's (7) & (8) $\Rightarrow$

$$\left(1 - \frac{\pi \lambda_1}{2}\right) c_1 = 0 \Rightarrow \left(1 - \frac{\pi}{2} \times \frac{2}{\pi}\right) c_1 = 0 \Rightarrow 0 c_1 = 0$$

$\Rightarrow c_1$ is arbitrary

and $\left(1 + \frac{\pi\lambda_1}{2}\right) c_2 = 0 \Rightarrow \left(1 + \frac{\pi}{2} \cdot \frac{2}{\pi}\right) c_2 = 0$

$$\Rightarrow 2 \cdot c_2 = 0 \Rightarrow c_2 = 0$$

put $c_2 = 0$ into (3), we have, $\because$ equ (3) is $\left[\because \phi(x) = \lambda c_1 \cos x - \lambda c_2 \sin x\right]$

$$\phi(x) = \lambda_1 c_1 \cos x$$

$\Rightarrow \phi(x) = \cos x$, if we take $\lambda_1 c_1 = 1$, $\lambda_1 = \frac{2}{\pi}$ (9)

which is the eigen vector corresponding to $\lambda_1 = \frac{2}{\pi}$

$$\therefore \Phi_1(x) = \frac{\cos x}{\|\cos x\|}, \quad \|\cos x\| = \sqrt{\int_0^{\pi} \cos^2 x \, dx} = \sqrt{\int_0^{\pi} \frac{1 + \cos 2x}{2} dx}$$

$$= \sqrt{\frac{\pi}{2}}$$

$\bar{\Phi}_1(x) = \sqrt{\frac{2}{\pi}} \cos x$ ——— (10)

$F_1 = \int_0^{\pi} f(x)\, \bar{\Phi}_1(x)\, dx$

$= \int_0^{\pi} 1 \cdot \sqrt{\frac{2}{\pi}} \cos x\, dx = \sqrt{\frac{2}{\pi}} \left[\sin x\right]_0^{\pi} = 0$

$F_1 = 0$. ——— (11)

To find eigen vector corresponding to $\lambda_2 = -\frac{2}{\pi}$

Equ's (7) and (8) $\Rightarrow$

$\left(1 - \frac{\pi\lambda_2}{2}\right) c_1 = 0 \Rightarrow \left[1 - \frac{\pi}{2}\left(-\frac{2}{\pi}\right)\right] c_1 = 0 \Rightarrow 2.c_1 = 0$
$\Rightarrow c_1 = 0$

$\left(1 + \frac{\pi\lambda_2}{2}\right) c_2 = 0 \Rightarrow \left[1 + \frac{\pi}{2}\left(-\frac{2}{\pi}\right)\right] c_2 = 0 \Rightarrow 0.c_2 = 0$
$\Rightarrow c_2$ is arbitrary

putting $c_1 = 0$ into (3), we get

i.e Equ (3) is

$\Phi(x) = -\lambda_2 c_2 \sin x \quad \left[\because \Phi(x) = \lambda c_1 \cos x - \lambda c_2 \sin x\right]$

$\Rightarrow \Phi_2(x) = \sin x$ if $-\lambda_2 c_2 = 1$, $\lambda_2 = -\frac{2}{\pi}$.

$\therefore \bar{\Phi}_2(x) = \frac{\sin x}{\|\sin x\|}$, $\|\sin x\| = \sqrt{\int_0^{\pi} \sin^2 x\, dx}$

$= \sqrt{\int_0^{\pi} \frac{1 - \cos 2x}{2}} = \sqrt{\frac{\pi}{2}}$

$\therefore \bar{\Phi}_2(x) = \sqrt{\frac{2}{\pi}} \sin x$ ——— (12)

$F_2 = \int_0^{\pi} f(x)\, \bar{\Phi}_2(x)\, dx$

$= \int_0^{\pi} 1 \cdot \sqrt{\frac{2}{\pi}} \sin x\, dx = \sqrt{\frac{2}{\pi}} (-) \left[\cos x\right]_0^{\pi}$

$F_2 = -\sqrt{\frac{2}{\pi}} \left[\cos \pi - \cos 0\right]$

$F_2 = 2\sqrt{\frac{2}{\pi}}$

$\therefore$ Required solu. of (1) is given by

$\Phi(x) = f(x) + c_1 \bar{\Phi}_1(x) + \frac{\lambda}{\lambda_2 - \lambda} F_2 \bar{\Phi}_2(x)$

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$$\phi(x) = 1 + c_1 \cdot \sqrt{\frac{2}{\pi}} \cos x + \frac{\lambda}{-\frac{2}{\pi} - \lambda} \cdot 2 \cdot \sqrt{\frac{2}{\pi}} \cdot \frac{\sin x}{\sqrt{\frac{\pi}{2}}}$$

[if $\lambda = \frac{2}{\pi}$, $\exists$ infinitely many solns.]

$$= 1 + c_1 \cdot \sqrt{\frac{2}{\pi}} \cos x + \frac{\frac{2}{\pi}}{-\frac{2}{\pi} - \frac{2}{\pi}} \cdot 2 \cdot \sin x \cdot \frac{2}{\pi}$$

$$\boxed{\phi(x) = 1 + c_1 \sqrt{\frac{2}{\pi}} \cos x - \frac{2}{\pi} \sin x}$$